NOTES ON THE DRUDE MODEL

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Abstract. Here are some notes about the classical theory of metals and the Drude model. The classical theory of electrons in metals is used to describe the basics of metallic behavior. These include their electrical conductivity, heat conductivity, reflection of optical wavelengths, etc.

1. Assumptions and basics

The Drude model was developed at the turn of the 20th century by Paul Drude. It came a few years after J.J. Thompson discovered the electron in 1897. It predates quantum theory, but still can tell us a lot about electrons in metals.

As background the the model we should get to know the electrons and how many we are dealing with. We're going to keep the valence assumption. This assumption rests on the intuition that the core electrons will be more tightly bound to their nuclei and hence will not be free to wander around and contribute to conduction. Essentially this lowers the number of electrons from Z to Z_c where Z_c is the number of conduction electrons.

So in a sample of metal say Sodium (Na) the density of conduction electrons, n is:

(1.1)
$$n = N_A \frac{Z_c \rho_m}{A} = 6.02 \times 10^{23} atoms/mol \frac{1e/atom \cdot 1 \times 10^6 g/m^3}{29 g/mol} = 2 \times 10^{28} e/m^3$$

where N_A is Avogadro's number, ρ_m is the density of the metal, A is the atomic number of the element and the numbers are for Na.

For the actual model we're going to eliminate all the electron ion interactions and replace them by a single parameter.

(1) We will treat collisions between e's and ions are instantaneous, uncorrelated events.
(2) We will ignore all other interactions (i.e. potentials from ions or other electrons) except applied fields. This means that electrons travel in straight lines between scattering events.
(3) Probability of an electron having a collision in a time interval dt is dt/τ. And τ does not depend on the electron position or momentum.
(4) Collisions 'thermalize' electrons. This means that after a collision the electrons have the temperature of the local environment.

2. Equations of motion

The first thing you need is to figure out how an electron's momentum, on average, will evolve over time. To do this we'll just find the average equation of motion for an electron. To find this let's start with the momentum of an electron at time t,

$\vec{p}(t)$, and find it at time $t + dt$. If the electrons had a collision it would on average have no momentum ($\vec{p}_c(t + dt) = 0$) at time $t + dt$ and by the third assumption above this has the probability, $P_c = dt/\tau$. This means that the probability of no collision is $P_{nc} = (1 - dt/\tau)$ this is because $P_c + P_{nc} = 1$. If there were no collision the electrons would have evolved normally and the electrons momentum becomes, $\vec{p}_{nc}(t + dt) = \vec{p}(t) + \vec{F}(t)dt$. This makes the new momentum:

$$\vec{p}(t + dt) = P_c \cdot \vec{p}_c(t + dt) + P_{nc} \cdot \vec{p}_{nc}(t + dt) = (1 - \frac{dt}{\tau})[\vec{p}(t) + \vec{F}(t)dt] \tag{2.1}$$

Using this to find the derivative take

$$\frac{\vec{p}(t)}{dt} = \frac{\vec{p}(t + dt) - \vec{p}(t)}{dt} = -\frac{\vec{p}(t)}{\tau} + \vec{F}(t) \tag{2.2}$$

And you have the equation of motion (EoM) averaged over electrons.

Of note there are a few regimes and solutions to consider.

- If $\vec{F}(t) = 0$ the solution to this homogeneous equation is $\vec{p}(t) = \vec{p}(0)e^{-t/\tau}$ which is why τ is called the relaxation time. If you impart momentum to the electrons on average they will relax back to no momentum exponentially with a time constant τ
- With a constant $\vec{F}$ you can show that the solution to the momentum $\vec{p}(t) = \vec{p}(0)e^{-t/\tau} + \vec{F}\tau$
- After a long time, $t \gg \tau$, the exponential term becomes negligible leaving $\vec{p}(t) = \vec{F}\tau$

3. Ohm's law

You all know Ohm's law:

$$V = IR \tag{3.1}$$

with V is the Voltage applied to a metal, I is the resulting current, and R is the proportionality constant. The main empirical fact here being the proportionality of current to the applied Voltage. This is what we're going to try to predict. But let's recast it in a form that is not dependent on the geometry of the experiment:

$$\vec{j} = \sigma\vec{E} \tag{3.2}$$

where σ is the conductivity of the metal and $\vec{j}$ is the current density. Since $\vec{j}$ is the current density it is the number of electrons passing a given point or $\vec{j} = 1/nev$ where n is still the electron density, e is still the electron charge, and v is the average drift velocity of the electrons.

In an applied electric field $\vec{E}$ the EoM for long times, $t \gg \tau$, gives us:

$$\vec{p}(t) = e\vec{E}\tau$$

$$\vec{v}(t) = \frac{e\vec{E}\tau}{m}$$

. Plugging this into the expression for the current density gives us

$$\vec{j} = \frac{ne^2\tau}{m}\vec{E} \tag{3.3}$$

which is Ohm's law with $\sigma = ne^2\tau/m$.

There are a couple of implications of this. For a metal like Na with a resistivity, $\rho_{Na} = 1/\sigma = 50$ nΩ·m the relaxation time is about 10^{-14} s.

4. AC conductivity and the shiny

In HW#1 you proved that a complex form of Ohm's law still holds for a metal in the Drude model with an oscillating electric field incident on a metal (i.e. $\vec{E}(t) = Re\{\vec{E}(\omega)e^{i\omega t}\}$). With an E-field of this form Ohm's law takes the form

$$\vec{j}(\omega) = \frac{ne^2\tau}{m}\frac{1}{i\omega\tau+1}\vec{E}(\omega)$$

where the ω dependent conductivity is given by

$$\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{i\omega\tau+1}$$

.

Since the electric field is oscillating electric we have an electromagnetic wave. So we should look at Maxwell's equations which with no net charge look like:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{d\vec{E}}{dt}$$

Since we have the complex Ohm's law we can replace the current density in the last equation and simplify it to be

$$\vec{\nabla}\times\vec{B} = (\mu_0\sigma(\omega) + i\omega\mu_0\epsilon_0)\vec{E}$$

Let's take the curl of the third equation and use the first one and what we just found to simplify it to

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}^2\vec{E} = \frac{d\vec{\nabla}\times\vec{B}}{dt} = i\omega(\mu_0\sigma(\omega) + i\omega\mu_0\epsilon_0)\vec{E}$$

After some rearranging this can be written as

$$-\vec{\nabla}^2\vec{E} = \frac{\omega^2}{c^2}(\frac{i\sigma(\omega)}{\omega\epsilon_0} - 1)\vec{E}$$

Now notice that if you take the limit where $\omega\tau \gg 1$, $\sigma(\omega) \to \sigma_0/i\omega\tau$ leaving us with

$$-\vec{\nabla}^2\vec{E} = \frac{\omega^2}{c^2}(\frac{ne^2}{m\epsilon_0\omega^2} - 1)\vec{E}$$

Now if you assume that the electric field has a plane wave spatial structure (i.e. $\vec{E}(t) = Re\{\vec{E}(\omega)e^{i\omega t}e^{i\vec{k}\cdot\vec{r}}\}\}$) then the above equation simplifies to the dispersion relation

$$k^2c^2 = (\omega^2 - \omega_p^2)$$

with $\omega_p = ne^2/m\epsilon_0$ being the plasma frequency.

This dispersion relationship tells us that k is imaginary for $\omega < \omega_p$ which is the result for an exponentially decaying electric field. This means that the EM wave doesn't propagate into a metal at these lower frequencies and instead is reflected. On the other hand for $\omega > \omega_p$ this gives a real k which is the form for a travelling wave. This implies that for frequencies higher than the plasma frequency metals become transparent. Which experiments verify (see figure).

For metals the plasma frequency is in the ultra violet and for Na it is about $\omega_p = 10^{15}\text{s}^{-1}$ which corresponds to a wavelength of light at λ 200nm.

Note: We disregard the direct magnetic field effects because they are much smaller than the electric field forces.

5. Joule heating

See the solution set for HW#1

6. Hall effect

We worked out what happened with a constant electric field put onto a metal. But what if you use a Voltage to drive current through a metal and put a perpendicular magnet field on it.

7. Thermal conductivity

The electrons in a metal aren't just good conductors of electricity, they are also good at conducting heat. So we're asking about the amount of heat that travels along a sample of metal as heat is applied. There is a simple phenomenological model that is analogous to Ohm's law, it is called Fourier's law and it looks like:

$$\vec{j}^q = -\kappa\vec{\nabla}T$$

The minus sign is because heat flows to lower temperature, and $\vec{j}^q$ is the heat current density $\Delta Q/\text{Time}\cdot A$, heat per time per area in units of [Watts/m^2] and κ is the thermal conductivity which has units of [Watts/mK]. So what does the Drude model predict?

A couple of assumptions are important here. First we need to remember the last assumption of the Drude model. This assumption means that after a collision an electron carries the thermal energy of the local environment. Also we need to assume that T varies a little over l (the mean free path electrons travel before scattering).

To calculate the heat delivered to a point we need to simplify this to a 1-D problem and then we need to calculate the heat it gets from the left and subtract the heat it gets from the right. Let's write the thermal energy an electron at temperature T has as $\mathcal{E}(T)$. Then at a point x the average electron coming from the left brings with it an energy $\mathcal{E}(T[x-v\tau])$ the average electron from the right delivers $\mathcal{E}(T[x+v\tau])$. So for electrons of density n and average velocity v (remembering that half will travel towards the point x and half away) this leaves us with

$$\vec{j}^q = \frac{1}{2}nv\{\mathcal{E}(T[x-v\tau] - \mathcal{E}(T[x+v\tau]\}$$

$$\vec{j}^q = \frac{1}{2}nv(\frac{d\mathcal{E}}{dT})(-\frac{dT}{dx})2v\tau$$

$$\vec{j}^q = nv^2\tau(\frac{d\mathcal{E}}{dT})(-\frac{dT}{dx})$$

Now note a few things $n\frac{d\mathcal{E}}{dT} = \frac{N}{V}\frac{d\mathcal{E}}{dT} = \frac{1}{V}\frac{dE}{dT} = c_V$ or the specific heat per volume. Also when we generalize to 3-D we note that $< v_x^2 >=< v_y^2 >=< v_z^2 >= \frac{1}{3} < v >^2$. Putting all this together we have

$$\vec{j}^q = \frac{< v >^2 \tau}{3}c_V(-\vec{\nabla}T)$$

So the prefactor up there is equal to the thermal conductivity.

There is another way to look at the specific heat. It's called the Wiedemann-Franz ratio. Which is given by

$$\frac{\kappa}{\sigma T} = \frac{m < v >^2 \tau c_V}{3e^2}$$

Now in a classical gas using the equipartition theorem the total thermal energy per volume is given by $E = n\frac{3}{2}k_B T$ so the specific heat per volume (the derivative of the energy with respect to energy) is $c_V = n\frac{3}{2}k_B$. And also note that the total thermal energy is the kinetic energy is given by $\frac{1}{2}m < v >^2 = n\frac{3}{2}k_B T$ which we can use to get rid of the $m < v >^2$ in the W-F ratio. All together this gives

$$\frac{\kappa}{\sigma T} = \frac{3}{2}(\frac{k_B}{e})^2 = 1.11 \times 10^{-8}(W\Omega/K^2)$$

This number only has fundamental constants in it and it is 1/2 the real value you get experimentally. But is really close for such a simple model.